

Game Theory: Logic, Set and Summation Notation

Branislav L. Slantchev

Department of Political Science, University of California - San Diego

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1 Formal Logic Refresher

Here's a notational refresher:

p, q, r		statements; either true or false, not both
$\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$		logical operators
$\neg p$	not p	"it is not the case that p "
$p \wedge q$	p and q	both p and q
$p \vee q$	p or q	either p or q or both
$p \Rightarrow q$	p implies q	if p then q
$p \Leftrightarrow q$	p if and only if q	p implies q and q implies p

We can build compound sentences using the above notation and then determine their truth or falsity with the use of the logical operators. The truth tables of the logical operators are very simple:

p	q	$p \wedge q$	p	q	$p \vee q$	p	q	$p \Rightarrow q$	p	$\neg p$
T	T	T	T	T	T	T	T	T	T	F
T	F	F	T	F	T	T	F	F	F	T
F	T	F	F	T	T	F	T	T	F	F
F	F	F	F	F	F	F	F	T	F	F

Note that the \vee operator is inclusive. That is, the statement $p \vee q$ is true if (a) only p is true, or (b) only q is true, or (c) both are true. This is different from the use of 'or' in everyday language, which is exclusive. That is, the statement p 'or' q is true if (a) only p is true, or (b) only q is true, but is false if both are true. Although sometimes a special symbol is used to denote the 'exclusive-or', this operator is redundant because we can simply write $[p \vee q] \wedge \neg[p \wedge q]$ to express it.

Be very careful where you place parentheses. For example, what does $p \vee q \Rightarrow \neg p$ mean? Does it mean $p \vee (q \Rightarrow \neg p)$ or does it mean $(p \vee q) \Rightarrow \neg p$? The first statement is always true while the second is true only when p is false.

We shall adopt the following convention. First, locate \neg followed by a simple statement such as p and take the brackets around p as implicit. Then, find instances where two statements are joined by \wedge or \vee and take the brackets around them as implicit. Thus, by this

convention, the correct interpretation of the example above is $(p \vee q) \Rightarrow (\neg p)$. This convention still does not allow us to interpret $p \wedge q \vee r$ unambiguously, and so we shall use brackets to make the meaning clear.

Pay special attention to the \Rightarrow operator because its expression in words may also be misleading. We can think of the statement $p \Rightarrow q$, as two separate statements. First, **sufficiency**: p is a sufficient condition for q . That is, whenever p is true, then q must be true as well, but whenever p is false, then q can be either true or false. Second, **necessity**: q is a necessary condition for p . That is, if q is false, then p must be false as well. If q is true, then p can be either true or false. You should make sure that you understand the following truth table which depicts p and q as jointly necessary and sufficient conditions.

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

We shall deal with necessary and sufficient conditions quite a bit, so you should make sure you thoroughly understand them.

A “tautology” is a compound statement that is always true (like $p \vee [q \Rightarrow \neg p]$, for example). A “theorem” (or proposition, or lemma or corollary) is a tautology of the form $[p_1 \wedge p_2 \wedge \dots \wedge p_k] \Rightarrow q$. The statements p_1, p_2, \dots, p_k are “assumptions.” We would generally write something like: “Assume (a) p_1 , (b) p_2 , ..., and (z) p_k . Then q .”

How do we prove theorems? There are several types of logical rules available:

1. *Simplification*: From $p \wedge q$ we can infer p and we can infer q .
2. *Addition*: From p we can infer $p \vee q$.
3. *Conjunction*: From p and q we can infer $p \wedge q$.
4. *Disjunctive Syllogism*: From $[p \vee q] \wedge \neg p$ we can infer q .
5. *Modus Ponens*: From $[p \Rightarrow q] \wedge p$ we can infer q .
6. *Modus Tollens*: From $[p \Rightarrow q] \wedge \neg q$ we can infer $\neg p$.
7. *Hypothetical Syllogism*: From $[p \Rightarrow q] \wedge [q \Rightarrow r]$ we can infer $[p \Rightarrow r]$.
8. *Constructive Dilemma*: From $[p \Rightarrow q] \wedge [s \Rightarrow t] \wedge [p \vee s]$ we can infer $[q \vee t]$.

To apply these rules, we usually need some rules of replacements (the ‘ \equiv ’ sign below reads “is interchangeable with,” which means “has the same truth value as”):

1. *Double Negation*: $p \equiv \neg \neg p$.
2. *Tautology*: $p \vee p \equiv p$.
3. *Commutation*: $p \vee q \equiv q \vee p$, and $p \wedge q \equiv q \wedge p$.
4. *Association*: $[p \vee q] \vee r \equiv p \vee [q \vee r]$, and $[p \wedge q] \wedge r \equiv p \wedge [q \wedge r]$.
5. *Distribution*: $p \vee [q \wedge r] \equiv [p \vee q] \wedge [p \vee r]$, and $p \wedge [q \vee r] \equiv [p \wedge q] \vee [p \wedge r]$.

6. *DeMorgan's Law*: $\neg[p \vee q] \equiv \neg p \wedge \neg q$, and $\neg[p \wedge q] \equiv \neg p \vee \neg q$.

7. *Contraposition*: $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$.

8. *Implication*: $p \Rightarrow q \equiv \neg p \vee q$.

The Rule of Conditional Proof: at some point in an argument we may want to deduce a conditional statement of the form $q \Rightarrow p$ from the statements already deduced. At this point, we can introduce q as an assumed premise. If we can then deduce p from q and the statements already deduced, then we can infer $q \Rightarrow p$.

For example, assume (a) $\neg q \vee p$, and (b) $\neg r \Rightarrow \neg p$. Prove $q \Rightarrow r$. The proof has several steps:

1. q assumed premise
2. $\neg\neg q$ double negation
3. p from (a), disjunctive syllogism
4. $\neg\neg p$ double negation
5. $\neg\neg r$ from (b), modus tollens
6. r double negation
7. $q \Rightarrow r$ rule of conditional proof

Thus, we started from q and the two assumptions, and arrived at r , thereby proving the statement.

An important variant of RCP is the *Proof by Contradiction* (one of my favorites). A “contradiction” is a compound statement of the form $q \wedge \neg q$. It is always false. The *Rule of Reductio ad Absurdum*, which is just a special case of RCP, is used when we want to deduce p . We introduce an assumed premise of the form $\neg p$ and try to deduce a contradiction from p and the statements already deduced.

For example, assume (a) $p \vee q$, (b) $q \Rightarrow [r \wedge w]$, and (c) $[r \vee p] \Rightarrow z$. Prove that z .

1. $\neg z$ assumed premise
2. $\neg[r \vee p]$ from (c), modus tollens
3. $\neg r \wedge \neg p$ DeMorgan's Law
4. $\neg p$ simplification
5. q from (a), disjunctive syllogism
6. $r \wedge w$ from (b), modus ponens
7. r simplification
8. $r \vee p$ addition
9. z from (c), modus ponens
10. $z \wedge \neg z$ from assumed premise, conjunction
11. z reductio ad absurdum

To see that RAA is just a special case of RCP, note that we take $\neg p$ as an assumed premise and deduce $q \wedge \neg q$. By RCP, we can infer $\neg p \Rightarrow q \wedge \neg q$. But $q \wedge \neg q$ is false regardless of the truth value of q . By modus tollens we can infer $\neg\neg p$, which by double negation is equivalent to p .

Sometimes we want to state generalities of the form “some element in the set X has a property $p(\cdot)$ ” and “every element in the set X has property $p(\cdot)$.” To do this, we use *quantifiers*:

- $(\exists x \in X)p(x)$ *existential* there exists an x in X such that $p(x)$
 $(\forall x \in X)p(x)$ *universal* for all x in X , $p(x)$

The precise meaning of the existential quantifier is the same as its English interpretation. However, what does the universal quantifier mean if the set X is empty? Thus, we clarify the interpretation to state “there is no x in X such that $p(x)$ is false”:

$$[(\forall x \in X)p(x)] \Leftrightarrow \neg[(\exists x \in X)\neg p(x)].$$

So, the original statement is vacuously (automatically) true if X is empty. For example, let X be the set of flying pigs, and let $p(x) =$ 'has wings'. Then $(\forall x \in X)p(x)$ is true. That is, it is not the case that there exists a flying pig that does not have wings. This is true because there exists no flying pig, with or without wings.

There are four rules of propositional logic:

1. *Universal Instantiation:* From $(\forall x \in X)p(x)$ we can infer $p(y)$ for every particular y in X .
2. *Existential Instantiation:* If $(\exists x \in X)p(x)$, then we can select a particular y in X and assume $p(y)$, as long as nothing else about y is assumed (it cannot appear previously in the proof).
3. *Existential Generalization:* If $p(y)$ holds for some (it does not matter which) particular y in X , we can infer $(\exists x \in X)p(x)$.
4. *Universal Generalization:* If we can prove $p(y)$ for some $y \in X$ without assuming anything about y other than its membership in X , we can infer $(\forall x \in X)p(x)$.

For example, assume (a) $(\exists x \in X)[p(x) \wedge q(x)]$, and (b) $(\forall x \in X)[q(x) \Rightarrow r(x)]$. Then $(\exists x \in X)[p(x) \wedge r(x)]$:

- | | |
|--|-------------------------------------|
| 1. $p(z) \wedge q(z)$ | from (a), existential instantiation |
| 2. $q(z) \Rightarrow r(z)$ | from (b), universal instantiation |
| 3. $q(z)$ | from (1), simplification |
| 4. $r(z)$ | from (2,3), modus ponens |
| 5. $p(z)$ | from (1), simplification |
| 6. $p(z) \wedge r(z)$ | from (4,5), conjunction |
| 7. $(\exists x \in X)[p(x) \wedge r(x)]$ | existential generalization |

Be careful when you apply universal generalization and existential instantiation. These can be tricky! Aristotle screwed it up and it took people over a thousand years to catch the mistake. Here’s what Aristotle said:

A.	all x are Y
B.	all x are Z
Therefore, some Y are Z	

Can you spot the mistake? Don’t worry, as I said, it took people a really long time to see it.¹

¹The conclusion does not follow if X is an empty set. So this is a problem with using the universal quantified improperly.

1 Set Theory Notation

We shall only review notation and some basic rules.

X, Y, Z	sets
x, y, z	elements of sets
$x \in X$	x is in X , or x is an element of X
$x \notin X$	x is not in X
$Y \subseteq X$	Y is a subset of X , or Y is included in X
$Y = X$	Y is equal to X , or $Y \subseteq X \wedge X \subseteq Y$
$Y \neq X$	Y is not equal to X , or $\neg[Y = X]$
$Y \subsetneq X$	Y is a proper subset of X , or $Y \subseteq X \wedge Y \neq X$
\emptyset	the empty set (set with no elements)

Note that the empty set is a subset of every set since $(\forall x \in X)(x \in X)$ is vacuously true for every X . There is only one empty set.

A set can be identified by listing its contents or by specifying a property common to all and only the elements of that set. For example, if X is the set of integers, then we can represent the set consisting of the number 1, 2, and 3 by

$$\{1, 2, 3\} \text{ or } \{x \in X \mid 0 < x < 4\}$$

where the second representation is read “the elements x of X such that $0 < x < 4$.” This method of defining sets is useful when the set is large. For example, if X is the set of real numbers, the set of positive real numbers is $\{x \in X \mid x > 0\}$. We could never list these numbers.

The following are some simple operations on sets. Let X be our “universe of discourse” (that is, everything happens in this set X). Let Y and Z be subsets of X . Then

1. *Intersection:* $Y \cap Z = \{x \in X \mid x \in Y \wedge x \in Z\}$
2. *Union:* $Y \cup Z = \{x \in X \mid x \in Y \vee x \in Z\}$
3. *Complement:* $\bar{Y} = \{x \in X \mid x \notin Y\}$
4. *Subtraction:* $Y \setminus Z = Y \cap \bar{Z} = \{x \in X \mid x \in Y \wedge x \notin Z\}$

The last operation is also called “the complement of Z relative to Y .”

DeMorgan’s Law (which you should recall from the previous section) is also applicable to sets. For example, let X be the universe of discourse. Then $\overline{[Y \cap Z]} = \bar{Y} \cup \bar{Z}$. To prove this, we need to show

$$(1) \overline{[Y \cap Z]} \subseteq \bar{Y} \cup \bar{Z} \quad \text{and} \quad (2) \bar{Y} \cup \bar{Z} \subseteq \overline{[Y \cap Z]}$$

We check (1). By the definition of \subseteq , we have to show that $(\forall x \in \overline{[Y \cap Z]})(x \in \bar{Y} \cup \bar{Z})$. Consider any $y \in \overline{[Y \cap Z]}$. By definition,

$$y \notin [Y \cap Z] \equiv \neg[y \in Y \wedge y \in Z] = \neg[y \in Y] \vee \neg[y \in Z],$$

where the last step is by DeMorgan’s Law for logical statements. We can rewrite the last statement as $y \notin Y \vee y \notin Z$, or $y \in \bar{Y} \vee y \in \bar{Z}$. Finally, by definition, we have $y \in \overline{[Y \cap Z]}$.

Since we assumed nothing about y except that it was one element of $\overline{[Y \cap Z]}$, we can use universal generalization to conclude that $(\forall x \in \overline{[Y \cap Z]})[x \in \overline{Y \cup Z}]$. This gives us (1). The proof for (2) is similar.

Sets are very useful for expressing order or duplication. For example, to represent a point on the Cartesian plane by a pair of numbers (one for horizontal and one for vertical position), we have to express order. Whenever order is important, we enclose the elements in parentheses: (5, 2). By convention, it is essential to list 5 before 2 because the point (5, 2) is quite different from the point (2, 5).

The elements do not have to be numbers: we can consider (x, y, z) where $x \in X$, $y \in Y$, and $z \in Z$, with X, Y , and Z being any sets at all. For example, let X be the set of numbers of Coke cans, Y be the set of numbers of whiskey shots, and Z be the set of numbers of wine glasses. Then (3, 1, 10) represents 3 Coke cans, 1 whiskey shot, and 10 glasses of wine. On the other hand, the element (10, 1, 3) represents 10 Coke cans, 1 shot of whiskey, and 3 glasses of wine.

When the elements are two, we call the signification of order an “ordered pair.” For more elements, we have ordered triples, all the way up to ordered n -tuples.

The set of ordered n -tuples from X_1, X_2, \dots, X_n is the “ n -fold cross product” (sometimes called the Cartesian product) written as $X_1 \times X_2 \times \dots \times X_n$. For example, when $n = 2$ the cross-product of X_1 and X_2 is defined as

$$X_1 \times X_2 = \{(x_1, x_2) | x_1 \in X_1 \wedge x_2 \in X_2\}$$

So, if $X_1 = \{1, 2, 3\}$ and $X_2 = \{\text{Mike, Suzie, Peter}\}$, then the cross-product is

$$X_1 \times X_2 = \{(1, \text{Mike}), (2, \text{Mike}), (3, \text{Mike}), \\ (1, \text{Suzie}), (2, \text{Suzie}), (3, \text{Suzie}), \\ (1, \text{Peter}), (2, \text{Peter}), (3, \text{Peter})\}$$

When $X_1 = X_2 = \dots = X_n = \mathbb{R}$, we refer to their cross product as the “ n -dimensional Euclidean space.” Sometimes it is useful to employ a shorthand notation for the cross-product:

$$\Delta X \equiv X_1 \times X_2 \times \dots \times X_n$$

Set theory is really something you should familiarize yourself with if you intend to do serious work in formal theory.

2 Summation Notation and Rules

We shall be using summation and multiplication quite often, so it’s worth refreshing our memories about the rules. We can condense summation and multiplication as follows:

$$x_1 + x_2 + x_3 + \dots + x_N = \sum_{i=1}^N x_i, \\ x_1 \times x_2 \times \dots \times x_N = \prod_{i=1}^N x_i.$$

Here are some useful rules:

1. For a constant a ,

$$\sum_{i=1}^N a = N \cdot a \quad \text{and} \quad \prod_{i=1}^N a = a^N$$

2. For a constant a and a variable x ,

$$\sum_{i=1}^N ax_i = a \cdot \sum_{i=1}^N x_i \quad \text{and} \quad \prod_{i=1}^N ax_i = a^N \cdot \prod_{i=1}^N x_i$$

3. For two variables x and y ,

$$\sum_{i=1}^N (x_i + y_i) = \sum_{i=1}^N x_i + \sum_{i=1}^N y_i$$

4. For two constants a and b , and two variables x and y ,

$$\sum_{i=1}^N (ax_i by_i) = a \cdot b \cdot \sum_{i=1}^N x_i y_i$$

Here are two mistakes that people sometimes make:

$$\sum_{i=1}^N x_i^2 \neq \left(\sum_{i=1}^N x_i \right)^2$$

and also

$$\sum_{i=1}^N x_i y_i \neq \sum_{i=1}^N x_i \cdot \sum_{i=1}^N y_i.$$

Sometimes we can employ shortcuts when the indexing is known, and so we would write

$$\sum ax_i$$

with the understanding that x is the variable, a is a constant, and the range of indexing is known. Sometimes (rarely) we would use double summation signs:

$$\sum_{i=1}^N \sum_{j=1}^K x_i y_j = \sum_{i=1}^N x_i \cdot \sum_{j=1}^K y_j = \sum_{j=1}^K \sum_{i=1}^N x_i y_j.$$

Note that x and y must be separately indexed for this to hold.