FINAL EXAMINATION

This is a closed book exam. You are only allowed to use class notes and lectures. You are not to consult with anyone or use Internet resources. ***Show all your work! I will take points off if you do not show me the derivations and all the steps.***

QUESTION 1. Two players play the following variant of the matching pennies game:

		Player 2	
		H	T
Player 1	H	$1+x_1, -1$	$-1, 1$
	T	$-1, 1+x_2$	$1, -1$

Figure 1: The Modified Matching Pennies Game.

Let x_1 and x_2 be independently drawn from the uniform distribution on $[0, x]$, and assume that each player observes his own draw but not the draw of the other player. Let p denote the probability that player 1 chooses H, and let q denote the probability that player 2 chooses T.

(i) What is player 1's best response to an arbitrary q? What is player 2's best response to an arbitrary p? (Hint: let $\hat{x}_i$ denote the type of player i that is precisely indifferent between H and T given the other player's mixing probability. This is the "cut-point" type for player i.)

(ii) Argue that in any Bayesian Nash equilibrium (BNE) of this game players use cut-point strategies. (That is, if some type of player i chooses H in BNE, then it must be that all types with higher realizations also choose H. Hint: try supposing that this is not true. You only need to show this for player 1.)

(iii) From player i's perspective, what is the equilibrium probability that player j chooses H (express this as a function of $\hat{x}_j$ using the assumption of a uniform distribution)?

(iv) Find the cut-points. (Hint: the step above gave you an expression for p as a function of $\hat{x}_1$ and an expression for q as a function of $\hat{x}_2$. You should end up with a system of equations where the cut-points are the two unknowns. You will need to solve a quadratic.)

(v) What is the equilibrium probability that player 1 chooses H as $x \to 0$? What is the equilibrium probability that player 2 chooses H as $x \to 0$? (Hint: you need to use the L'Hôpital Rule.) Comment on this finding.

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QUESTION 2. Suppose a prospector (player 1) owns a gold mine with x amount of gold in it. He can work the mine himself or he can sell it to a company (player 2). If the prospector works the mine, the payoffs are $(3x, 0)$. If the company buys the mine at some price $\pi \geq 0$, the payoffs are $(\pi, 4x - \pi)$. The prospector knows x but the company does not.

For the following questions, consider a variant of this game where x can take one of three values: $x \in \{1/4, 1/2, 3/4\}$, each with equal probability. Let the "type" of player 1 be denoted by L, M, and H to correspond to these amounts. Player 1 can opt to mine himself or try to sell the mine. If he tries to sell, he makes a take-it-or-leave it offer at some price π, and player 2 can either accept it (in which case she buys the mine to work it) or reject it (in which case player 1 works the mine).

(i) Suppose the amount of gold is common knowledge. Will trade occur? (Answer this for each of the three possible values.)

(ii) Construct a pure-strategy Nash equilibrium when player 1's type is commonly known to be M and where trade occurs at $\pi = 7/4$. Is it subgame-perfect? If not, find the subgame-perfect equilibria for this case.

(iii) Consider now the case where only player 1 knows the value of x. Can there be a perfect Bayesian equilibrium (PBE) in which trade occurs at several different prices?

(iv) Is there a pre-strategy PBE in which all three types of player 1 trade at some price π? (Hint: write player 2's expected payoff from buying at this price and note that in equilibrium she must not be better off by refusing to buy.)

(v) Is there a PBE in which only the L and M types trade at some price π?

For the following questions, consider a variant of this game where player 1's type is drawn from the uniform distribution on [0, 1], and only player 1 observes the value of x.

(vi) Can there be a PBE in which trade occurs at different prices? Why or why not?

(vii) Consider a fixed price π. Which types would sell at this price?

(viii) If in equilibrium player 1 only tries to sell the mine when he expects to sell it and the equilibrium price is π, what is player 2's belief upon observing π?

(ix) What is player 2's expected payoff from buying at this price? Will she buy at this price? What can you conclude about the probability of trade in PBE in this case?

For the following question, consider a variant of this game where x is distributed uniformly on [0, 1], only player 1 knows the value of x, but it is player 2 who makes a take-it-or-leave-it offer at price π, which player 1 can either accept (in which case player 2 buys the mine to work it) or reject (in which case player 1 works the mine).

(x) Derive player 1's best response to some price π offered by player 2.

(xi) What is player 2's expected payoff from offering some π? What can you conclude about the probability of trade in PBE in this case? Is this payoff different from what you found in (ix)? Why is that?