PURE AND MIXED STRATEGY EQUILIBRIA. SUBGAME PERFECTION

Write your answers neatly on a separate sheet of paper (attach as many sheets as necessary). Show all calculations (no credit for answers giving final result only). Justify your steps. Remember to write your name and staple all pages together.

QUESTION 1. THE SUPPLY OF BOUGAINVILLE. During World War II the U.S. invaded the Solomon Islands. There was a period when U.S. forces had occupied some of the southern islands but not yet captured the northwestern island of Bougainville. A U.S. admiral had to maintain supply routes against Japanese defenses. U.S. ships had two routes that they could take for resupply. The northern route was quicker and less costly than the southern route. All other things equal, the Americans would prefer to go north. The enemy had enough ships to block successfully only one route. Figure 1 gives the payoff matrix for this game.

		Americans: Sail North (N)	Americans: Sail South (S)
Japanese	Defend North (N)	$(200, -100)$ short route; supply ships sunk	$(-100, 50)$ supply slow; forces rearmed
	Defend South (S)	$(-100, 125)$ supply fast; forces rearmed	$(200, -175)$ costly route; supply ships sunk

Figure 1: Supply of Bougainville.

(a) What are the Nash equilibria in pure strategies?

(b) What would happen to the Americans if they picked a pure strategy? To the Japanese?

(c) Let p be the probability that the Americans pick N, and let q be the probability that the Japanese pick N. Find the mixed strategy Nash equilibrium.

(d) What are the expected utilities of the Americans and the Japanese for picking each of the routes given their equilibrium strategies?

(e) What is the probability that the supply ships would get sunk in the mixed strategy equilibrium? (Hint: to calculate the probability that both players choose the same route, e.g. N, multiply the individual probabilities. That is, if p is the probability that the Americans choose N and q is the probability that the Japanese choose N, then the probability that they both choose N is $p \times q$. The probability that they both choose S is $(1-p) \times (1-q)$. To calculate the probability that either one of two events will occur, add their individual probabilities. For example, if r is the probability of event A and s is the probability of event B, then the probability of either A or B is $r + s$.)

Question 2. Battle of the Sexes. This is a game that characterizes common situations involving problems with coordination and the distribution of resources. The original (politically incorrect) story goes like this.[1] A man (the row player) and a woman (the column player) each have two choices for an evening's entertainment. Each can either go to a prize fight, F, or to a ballet, B. Following the usual cultural stereotype, the man much prefers the fight and the woman the ballet; however, to both it is more important that they go out together than that each see the preferred entertainment.

The payoff matrix in Figure 2 represents this situation but with an additional twist. The man is the modern "sensitive" type and feels psychic discomfort from liking rough entertainment. Therefore, he pays an additional cost $c > 0$ every time he chooses the prize fight.

		Woman: Prize Fight (F)	Woman: Ballet (B)
Man	Prize Fight (F)	$(3-c, 2)$ together at the prize fight	$(-c, 0)$ man at fight, woman at ballet
Man	Ballet (B)	$(0, 0)$ man at ballet, woman at fight	$(2, 3)$ together at the ballet

Figure 2: The Modern Battle of the Sexes.

(a) Let p be the probability that the man chooses F, and let q be the probability that the woman chooses B. Write the man's expected utilities from chooses F and B in terms of the woman's mixing probability q.

(b) Let $\underline{c}$ denote the critical cost such that the man always prefers to choose B whenever $c > \underline{c}$. Find $\underline{c}$. What is the Nash equilibrium if $c > \underline{c}$?

(c) Suppose $c < \underline{c}$. Find the mixed strategy Nash equilibrium.

(d) Using the result in (c), what are the players' equilibrium mixing probabilities if $c = 0.5$?

(e) For your answer in (d), what is the equilibrium probability that both players will go to the fight? The ballet? Fail to coordinate? (Hint: refer to the hint for part (e) in Question 1.)

(f) If you had to make a prediction based on this model about the likely outcomes in a modern society ($c > \underline{c}$) and a traditional one ($c = 0$), what would it be?

[1] Taken almost verbatim from Luce & Raiffa, *Games and Decisions*, p. 91.

Question 3. General Deterrence. North Korea is considering whether to challenge U.S. interests in the Far East by developing capability to produce nuclear weapons. If it challenges the status quo and builds facilities for enrichment of weapons-grade uranium, the United States has to decide whether to acquiesce or make a deterrent threat. If it backs down, the North Koreans end with a successful challenge and become militarily stronger. If it issues the threat, the Koreans can either back down or press ahead with their program. If they back down, their bluff is called and they suffer a loss in prestige. If they press ahead, the United States must decide whether to carry out the threat, in which case an armed conflict results, or renege on its commitment, in which it suffers a loss of prestige and the challenge succeeds.

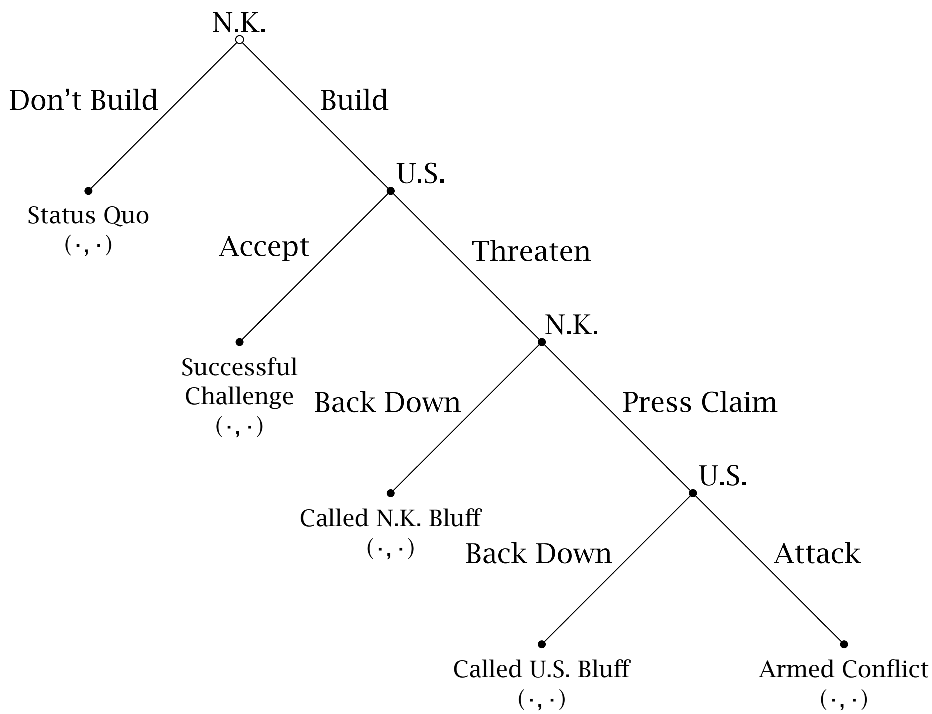

Figure 3: General Deterrence of North Korean Nuclear Program.

For the questions that follow, associate each outcome with an ordinal value from 1 to 5, with 1 denoting the worst outcome, and 5 denoting the best.

(a) Assume the following preferences:

- N.K.: Successful Challenge $\succ$ Called U.S. Bluff $\succ$ Status Quo $\succ$ Called N.K. Bluff $\succ$ Armed Conflict
- U.S.: Status Quo $\succ$ Called N.K. Bluff $\succ$ Armed Conflict $\succ$ Successful Challenge $\succ$ Called U.S. Bluff

Write the payoffs for each outcome. (The first number is N.K.'s payoff and the second is U.S.'s.) Find the subgame perfect equilibrium of this game. Explain the outcome.

(b) Keep the North Korean preferences as in (a), and assume the following:

- U.S.: Status Quo $\succ$ Called N.K. Bluff $\succ$ Successful Challenge $\succ$ Called U.S. Bluff $\succ$ Armed Conflict

Write the new payoffs for the two players for each outcome. Find the subgame perfect equilibrium of this game. Explain why this equilibrium occurs.

QUESTION 4. BRIBING NORTH KOREA. In the example given in class we *assumed* that if the North Koreans accept the bribe they dismantle the plant. This is not realistic. Let's now assume that once they accept the bribe, they can choose whether to dismantle the plant or not. If they don't the United States can (again) take it out with an air strike. The Koreans get a bonus of international goodwill worth \$1 million if they negotiate (that is, if they do not reject an offer) even if they do so in bad faith. Assume that the minimum offer the United States can make is \$1 million (otherwise the North Koreans would not even negotiate).

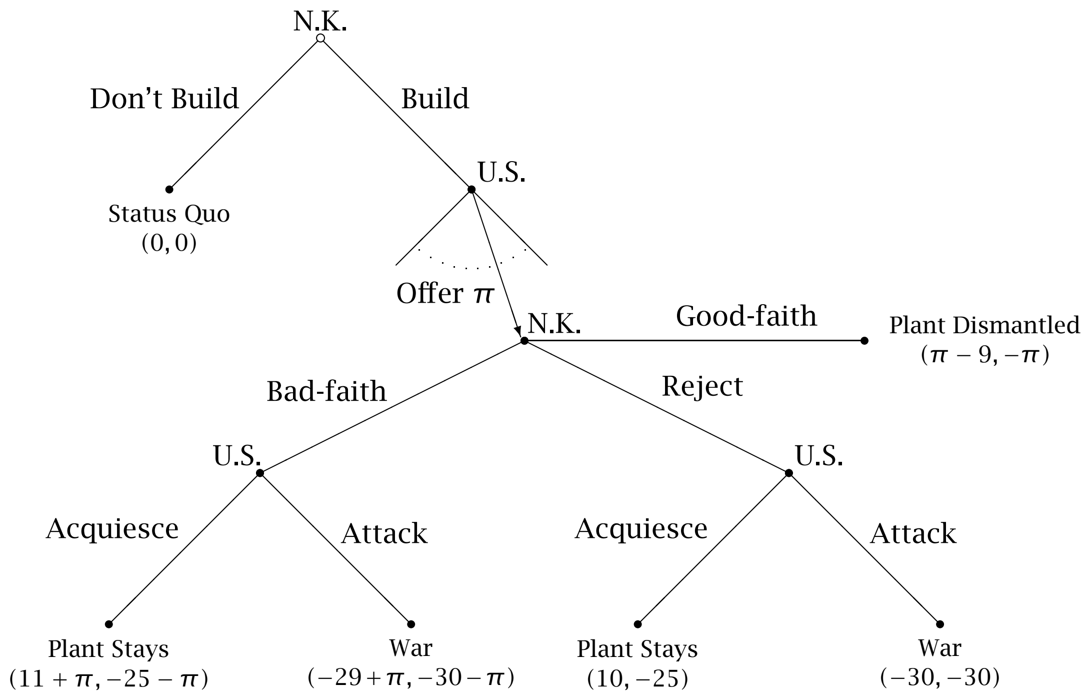

Figure 4: Bribing the North Koreans.

(a) What is the outcome in the subgame perfect equilibrium of this game?

(b) How does that differ from the outcome we arrived at in class? Why?