PURE AND MIXED STRATEGY EQUILIBRIA. SUBGAME PERFECTION

Write your answers neatly on a separate sheet of paper (attach as many sheets as necessary). Show all calculations (no credit for answers giving final result only). Justify your steps. Remember to write your name and staple all pages together.

QUESTION 1. THE SUPPLY OF BOUGAINVILLE. During World War II the U.S. invaded the Solomon Islands. There was a period when U.S. forces had occupied some of the southern islands but not yet captured the northwestern island of Bougainville. A U.S. admiral had to maintain supply routes against Japanese defenses. U.S. ships had two routes that they could take for resupply. The northern route was quicker and less costly than the southern route. All other things equal, the Americans would prefer to go north. The enemy had enough ships to block successfully only one route. Figure 1 gives the payoff matrix for this game.

		Americans: Sail North (N)	Americans: Sail South (S)
Japanese	Defend North (N)	$(200, -100)$ short route; supply ships sunk	$(-100, 50)$ supply slow; forces rearmed
	Defend South (S)	$(-100, 125)$ supply fast; forces rearmed	$(200, -175)$ costly route; supply ships sunk

Figure 1: Supply of Bougainville.

(a) What are the Nash equilibria in pure strategies?

(b) What would happen to the Americans if they picked a pure strategy? To the Japanese?

(c) Let p be the probability that the Americans pick N, and let q be the probability that the Japanese pick N. Find the mixed strategy Nash equilibrium.

(d) What are the expected utilities of the Americans and the Japanese for picking each of the routes given their equilibrium strategies?

(e) What is the probability that the supply ships would get sunk in the mixed strategy equilibrium? (Hint: to calculate the probability that both players choose the same route, e.g. N, multiply the individual probabilities. That is, if p is the probability that the Americans choose N and q is the probability that the Japanese choose N, then the probability that they both choose N is $p \times q$. The probability that they both choose S is $(1 - p) \times (1 - q)$. To calculate the probability that either one of two events will occur, add their individual probabilities. For example, if r is the probability of event A and s is the probability of event B, then the probability of either A or B is $r + s$.)

Answer. This is a simple strategic form game and we can find the pure strategy Nash equilibria by inspection of the payoff matrix. We can find all Nash equilibria by analyzing best responses, as usual.

(a) There are no Nash equilibria in pure strategies. For every possible pair of actions, one player prefers to deviate to another action.

(b) If the Americans pick any pure strategy, the Japanese would respond with their optimal choice, which leaves the Americans strictly worse off. For example, if the Americans sail north, the Japanese would respond by defending north and sinking the U.S. ships. If the Americans sail south, the Japanese would defend the south and again sink the ships. The reverse is true if the Japanese defend either route with certainty: the Americans would pick the other and their ships would get through.

(c) In a mixed strategy equilibrium players are indifferent between their pure strategies. That is, the expected utilities of playing the pure strategies are equal:

$$\begin{aligned} EU_A(N) &= EU_A(S) \\ -100q + 125(1-q) &= 50q - 175(1-q) \\ 125 - 225q &= 225q - 175 \\ 450q &= 300 \\ q &= \frac{2}{3} \end{aligned} \qquad \begin{aligned} EU_J(N) &= EU_J(S) \\ 200p - 100(1-p) &= -100p + 200(1-p) \\ 300p - 100 &= -300p + 200 \\ 600p &= 300 \\ p &= \frac{1}{2} \end{aligned}$$

We conclude that the mixed strategy equilibrium is the pair of strategies where the Americans sail north and south with equal probability of 0.5, and the Japanese defend the north route with probability 2/3 and defend the south route with probability 1/3.

(d) We plug in the probabilities we found in the expressions for the expected utilities:

$$EU_A(N) = -100\left(\frac{2}{3}\right) + 125\left(\frac{1}{3}\right) = -25 \qquad EU_J(N) = 200\left(\frac{1}{2}\right) - 100\left(\frac{1}{2}\right) = 50$$

$$EU_A(S) = 50\left(\frac{2}{3}\right) - 175\left(\frac{1}{3}\right) = -25 \qquad EU_J(S) = -100\left(\frac{1}{2}\right) + 200\left(\frac{1}{2}\right) = 50$$

Of course, as expected, both players receive the same utilities from each of their two actions.

(e) The probability that the ships would get sunk is the sum of the probabilities that both players pick the same route, either the north or the south one. The probability that both the Japanese and the Americans pick the same route is the product of the probabilities with which each player picks that route:

$$\Pr(\text{both pick } N) = p \times q = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$$

$$\Pr(\text{both pick } S) = (1-p) \times (1-q) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

Therefore, the probability that the ships get sunk in the mixed strategy equilibrium is: $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$.

Question 2. Battle of the Sexes. This is a game that characterizes common situations involving problems with coordination and the distribution of resources. The original (politically incorrect) story goes like this.[1] A man (the row player) and a woman (the column player) each have two choices for an evening's entertainment. Each can either go to a prize fight, F, or to a ballet, B. Following the usual cultural stereotype, the man much prefers the fight and the woman the ballet; however, to both it is more important that they go out together than that each see the preferred entertainment.

The payoff matrix in Figure 2 represents this situation but with an additional twist. The man is the modern "sensitive" type and feels psychic discomfort from liking rough entertainment. Therefore, he pays an additional cost $c > 0$ every time he chooses the prize fight.

[1] Taken almost verbatim from Luce & Raiffa, *Games and Decisions*, p. 91.

		Woman	
		Prize Fight (F)	Ballet (B)
Man	Prize Fight (F)	$(3-c, 2)$ together at the prize fight	$(-c, 0)$ man at fight, woman at ballet
	Ballet (B)	$(0, 0)$ man at ballet, woman at fight	$(2, 3)$ together at the ballet

Figure 2: The Modern Battle of the Sexes.

(a) Let p be the probability that the man chooses F, and let q be the probability that the woman chooses B. Write the man's expected utilities from choosing F and B in terms of the woman's mixing probability q.

(b) Let $\underline{c}$ denote the critical cost such that the man always prefers to choose B whenever $c > \underline{c}$. Find $\underline{c}$. What is the Nash equilibrium if $c > \underline{c}$?

(c) Suppose $c < \underline{c}$. Find the mixed strategy Nash equilibrium.

(d) Using the result in (c), what are the players' equilibrium mixing probabilities if $c = 0.5$?

(e) For your answer in (d), what is the equilibrium probability that both players will go to the fight? The ballet? Fail to coordinate? (Hint: refer to the hint for part (e) in Question 1.)

(f) If you had to make a prediction based on this model about the likely outcomes in a modern society ($c > \underline{c}$) and a traditional one ($c = 0$), what would it be?

Answer. We do each part in turn (always a good idea).

(a) The expected utilities for the man are

$$EU_M(F) = (3-c)(1-q) + (-c)q = 3 - c - 3q$$
$$EU_M(B) = 0(1-q) + 2q = 2q$$

and the expected utilities for the woman are

$$EU_W(F) = 2p + 0(1-p) = 2p$$
$$EU_W(B) = 0p + 3(1-p) = 3 - 3p$$

(b) The man prefers to choose B if its expected utility exceeds the expected utility from choosing F:

$$EU_M(B) \geq EU_M(F)$$
$$2q \geq 3 - c - 3q$$
$$c \geq 3 - 5q$$

Since the man always prefers B, we pick the largest cost which is when $q = 0$:

$$c \geq 3 - 5(0)$$
$$c \geq 3$$
$$\Rightarrow \underline{c} = 3$$

In words, whenever $c > 3$ (that is $c > \underline{c}$), the man would pick B regardless of what the woman does.In this case the woman also always chooses B, and so (B, B) is the unique pure-strategy Nash equilibrium.

You can see this by picking some $c > 3$, e.g. $c = 4$, and filling in the payoff matrix. Suppose the woman goes to the fight. If the man goes to the fight, he gets $3 - 4 = -1 < 0$ which he would get if he goes to the ballet. So he would go to the ballet. Suppose the woman goes to the ballet. If the man goes to the fight, he gets $-4 < 2$, which he would get if he goes to the ballet. So he would go to the ballet.

Intuitively, if the man's costs to choosing the fight are sufficiently high, he will not choose the fight but go to the ballet instead regardless of what the woman does (the expression above holds for all q). If he goes to the ballet, then the woman's best response is to go to the ballet as well. A modern man and a woman will always end up at the ballet.

(c) If $0 < c < 3$, then going to the ballet no longer strictly dominates going to the fight for the man. The man prefers to go to the ballet whenever the woman goes to the ballet, and he prefers to go to the fight whenever the woman goes to the fight. We have two equilibria in pure strategies, (F, F) and (B, B). To find the equilibrium in mixed strategies, set each player's expected utilities equaling each other, as before:

$$\begin{aligned} EU_M(F) &= EU_M(B) \\ 3 - c - 3q &= 2q \\ 5q &= 3 - c \\ q &= \frac{3-c}{5} \end{aligned} \qquad\qquad \begin{aligned} EU_W(F) &= EU_W(B) \\ 2p &= 3 - 3p \\ 5p &= 3 \\ p &= \frac{3}{5} \end{aligned}$$

Since $0 < c < 3$, $q = (3 - c)/5$ is a valid probability, so we're fine. The mixed strategy Nash equilibrium is for the man to go to the fight with probability $3/5$, and the woman to go to the ballet with probability $(3 - c)/5$. The exact probability, of course, depends on the cost magnitude.

(d) All we have to do is now plug in the value for c in the expression for the woman's mixing probability (it is the only one that involves the cost parameter). We get

$$q = \frac{3-c}{5} = \frac{3-0.5}{5} = 0.5$$

Therefore, the mixing probabilities are $3/5$ for the man and $1/2$ for the woman. Let's check our calculations. If the man chooses F, his expected payoff is $(3 - 0.5)(0.5) + (-0.5)(0.5) = 1$, and if he chooses B, his expected payoff is $(0)(0.5) + (2)(0.5) = 1$, and so we're fine. If the woman chooses F, her expected payoff is $(2)(3/5) + (0)(2/5) = 6/5$, and if she chooses B, her expected payoff is $(0)(3/5) + (3)(2/5) = 6/5$, and so we're fine.

(e) We have the exact probabilities from (d). The probability that both players go to the fight is just the product of the individual probabilities for each going to the fight:

$$\Pr(\text{both go to the fight}) = p \times (1 - q) = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}.$$

The probability that both players go to the ballet is also the product of the corresponding individual probabilities:

$$\Pr(\text{both go to the ballet}) = (1 - p) \times q = \frac{2}{5} \times \frac{1}{2} = \frac{1}{5}.$$

The probability that they fail to coordinate is the sum of the probabilities that they go to different places:

$$\begin{aligned} &\Pr(\text{fail to coordinate}) \\ &= \Pr(\text{man goes to ballet and woman to the fight}) + \Pr(\text{man goes to the fight and woman to the ballet}) \\ &= (1 - p) \times (1 - q) + p \times q \\ &= \frac{2}{5} \times \frac{1}{2} + \frac{3}{5} \times \frac{1}{2} = \frac{1}{5} + \frac{3}{10} = \frac{1}{2} \end{aligned}$$

Of course, you could have gotten the same result by finding the probability that they will coordinate (it's the sum of the probabilities that they end up both at the fight or the ballet, which you have already calculated) and then subtracting the result from 1. You get the same: $1 - \left(\frac{3}{10} + \frac{1}{5}\right) = 1 - 0.5 = 0.5$, as expected.

(f) The prediction for the modern society would be based on the high costs, in which case (as we found out above) the man always goes to the ballet, to which the woman best-responds by going to the ballet as well.

In a modern society, ballet would edge out fighting as the preferred form of entertainment because men and women would mostly coordinate on going to ballet. Moreover, player never fail to coordinate in that society.

In a traditional society, on the other hand, both forms of entertainment are likely to persist because we have two equilibria in pure strategies. In the mixed equilibrium, players sometimes coordinate on ballet, sometimes on the fight, and sometimes do not coordinate at all.

QUESTION 3. GENERAL DETERRENCE. North Korea is considering whether to challenge U.S. interests in the Far East by developing capability to produce nuclear weapons. If it challenges the status quo and builds facilities for enrichment of weapons-grade uranium, the United States has to decide whether to acquiesce or make a deterrent threat. If it backs down, the North Koreans end with a successful challenge and become militarily stronger. If it issues the threat, the Koreans can either back down or press ahead with their program. If they back down, their bluff is called and they suffer a loss in prestige. If they press ahead, the United States must decide whether to carry out the threat, in which case an armed conflict results, or renege on its commitment, in which it suffers a loss of prestige and the challenge succeeds.

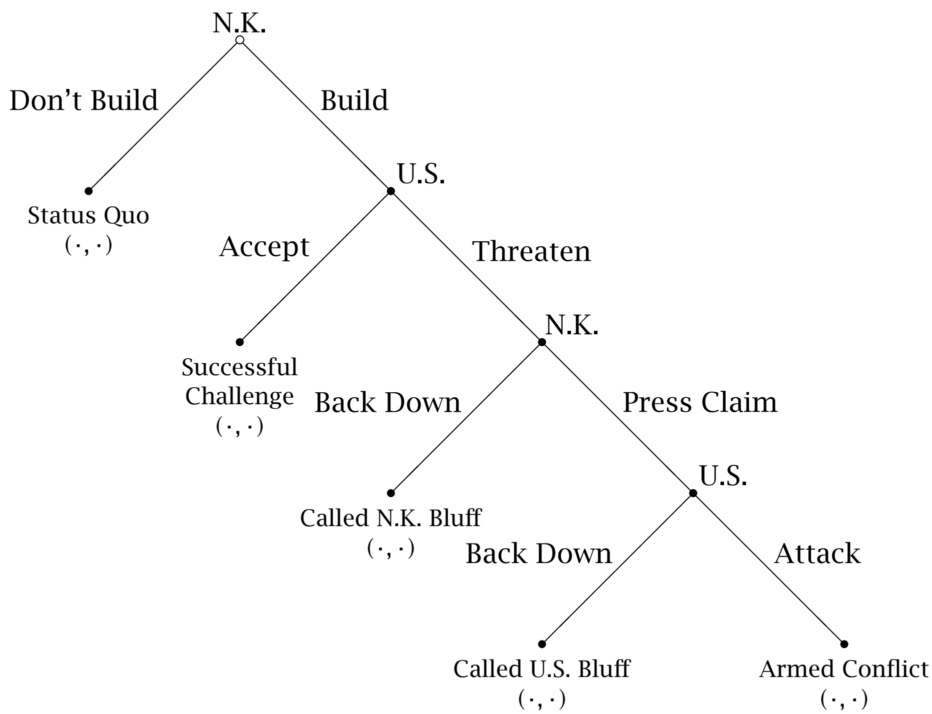

Figure 3: General Deterrence of North Korean Nuclear Program.

For the questions that follow, associate each outcome with an ordinal value from 1 to 5, with 1 denoting the worst outcome, and 5 denoting the best.

(a) Assume the following preferences:

- N.K.: Successful Challenge $\succ$ Called U.S. Bluff $\succ$ Status Quo $\succ$ Called N.K. Bluff $\succ$ Armed Conflict
- U.S.: Status Quo $\succ$ Called N.K. Bluff $\succ$ Armed Conflict $\succ$ Successful Challenge $\succ$ Called U.S. Bluff

Write the payoffs for each outcome. (The first number is N.K.'s payoff and the second is U.S.'s.) Find the subgame perfect equilibrium of this game. Explain the outcome.

(b) Keep the North Korean preferences as in (a), and assume the following:

- U.S.: Status Quo $\succ$ Called N.K. Bluff $\succ$ Successful Challenge $\succ$ Called U.S. Bluff $\succ$ Armed Conflict

Write the new payoffs for the two players for each outcome. Find the subgame perfect equilibrium of this game. Explain why this equilibrium occurs.

Answer. We first write the numbers, then fill in the blanks, and then solve the resulting game.

(a) The payoffs given the preference orderings:

N.K.		U.S.	
Successful Challenge:	5	Successful Challenge:	2
Called U.S. Bluff:	4	Called U.S. Bluff:	1
Status Quo:	3	Status Quo:	5
Called N.K. Bluff:	2	Called N.K. Bluff:	4
Armed Conflict:	1	Armed Conflict:	3

The game tree with the payoffs filled in:

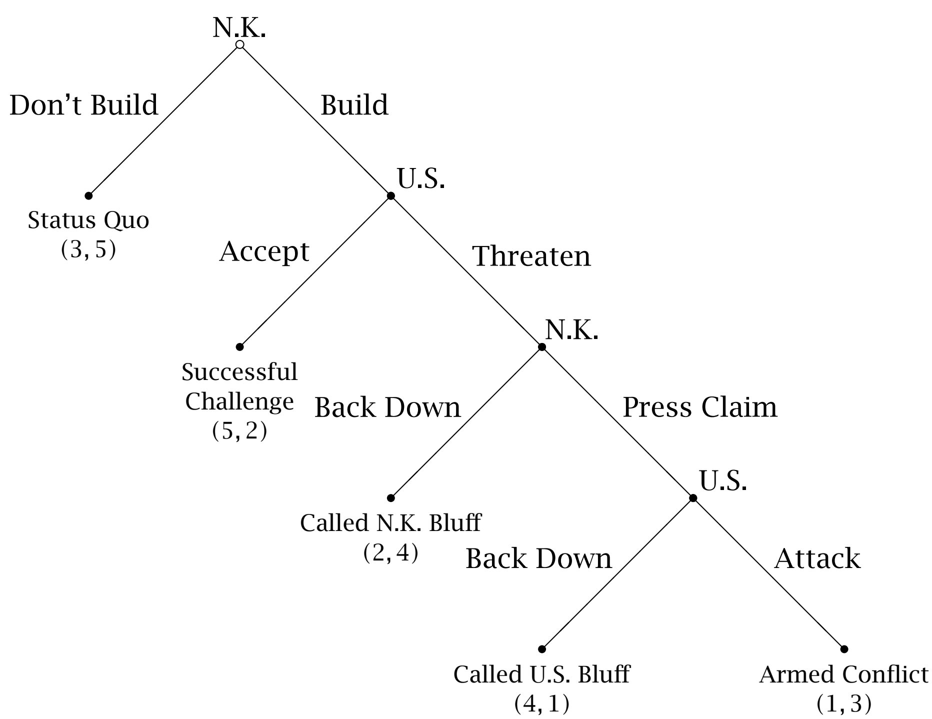

Figure 4: General Deterrence of North Korean Nuclear Program, With Payoffs.

We find the subgame perfect equilibrium by backward induction. The last to move is the U.S., which strictly prefers attacking to backing down (gets 3 in the first case versus 1 in the other). Given that the U.S. is going to attack, N.K. prefers backing down to pressing the claim, giving it a payoff of 2 instead of 1. Since N.K.

would back down if threatened, the U.S. prefers to threaten (payoff of 4) to accepting (payoff of 2). Finally, given that the U.S. is going to threaten and that N.K. will back down after the threat, N.K. prefers not to build the plant in the first place (payoff of 3) to building it (payoff of 2).

Therefore, the subgame perfect equilibrium is the pair of strategies where the North Koreans don't build and back down when threatened, and the Americans threaten, and attack if the North Koreans press their claim. Note again that the strategies are complete contingent plans of actions. In particular, the North Korean strategy specifies what to do if they build and the U.S. threatens even though the same strategy specifies that they should not build. Again, this is necessary to have in order to evaluate the optimality of U.S. behavior, which in turn is used to evaluate the optimality of North Korean behavior.

The outcome is that the North Koreans never challenge the status quo (don't build the plant). It occurs because the U.S. has a credible threat to attack should the North Koreans ignore the threat and press with their claim.

(b) The payoffs are the same for North Korea, but we change them for the U.S.:

U.S.	Successful Challenge:	3
	Called U.S. Bluff:	2
	Status Quo:	5
	Called N.K. Bluff:	4
	Armed Conflict:	1

The game tree with the payoffs filled in:

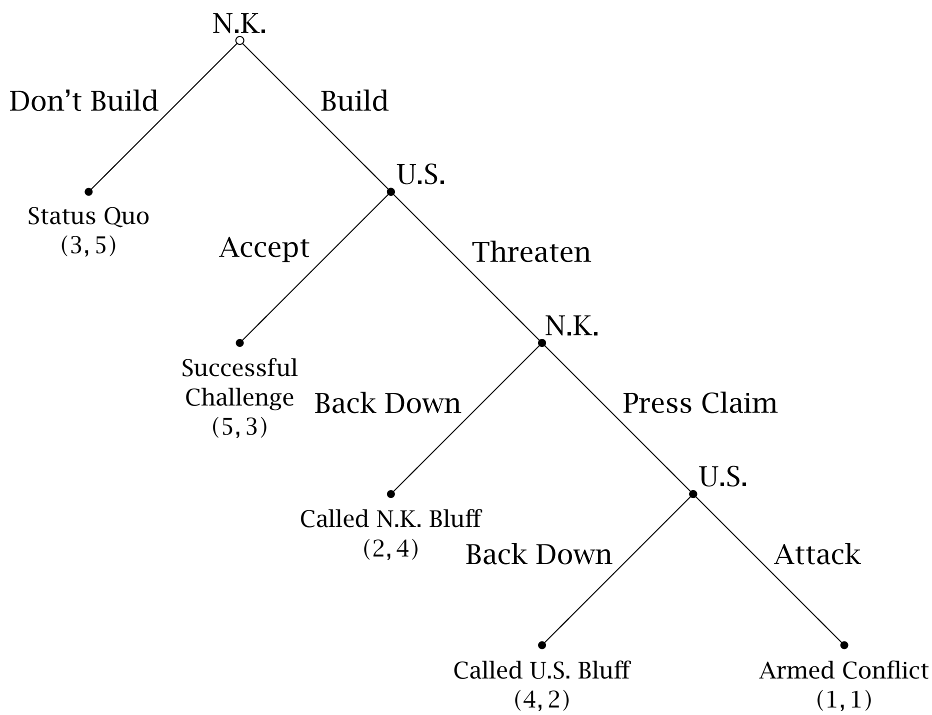

Figure 5: General Deterrence of North Korean Nuclear Program, With Payoffs, II.

We find the subgame perfect by backward induction. At the last move, the U.S. backs down. Given that, N.K. presses the claim. Given that, the U.S. accepts the challenge instead of threatening. Given that, N.K. builds the plant. The subgame perfect equilibrium pair of strategies is for N.K. to build the plant, and press the claim if the U.S. threatens; and for U.S. it is to accept the challenge, and back down if N.K. presses the claim.

This equilibrium occurs because the U.S. does not have a credible threat to attack if the North Koreans press their claim following a threat. Since this threat is incredible, N.K. would press the claim, which means that for the U.S. it is preferable to accept the initial challenge than to threaten incredible reprisals. Since the U.S. accommodates, N.K. challenges the status quo. The outcome is that N.K. builds a plant and the U.S. accepts it. Contrast this with the first outcome, where the U.S. did have a credible military threat at its disposal.

QUESTION 4. BRIBING NORTH KOREA. In the example given in class we *assumed* that if the North Koreans accept the bribe they dismantle the plant. This is not realistic. Let's now assume that once they accept the bribe, they can choose whether to dismantle the plant or not. If they don't the United States can (again) take it out with an air strike. The Koreans get a bonus of international goodwill worth \$1 million if they negotiate (that is, if they do not reject an offer) even if they do so in bad faith. Assume that the minimum offer the United States can make is \$1 million (otherwise the North Koreans would not even negotiate).

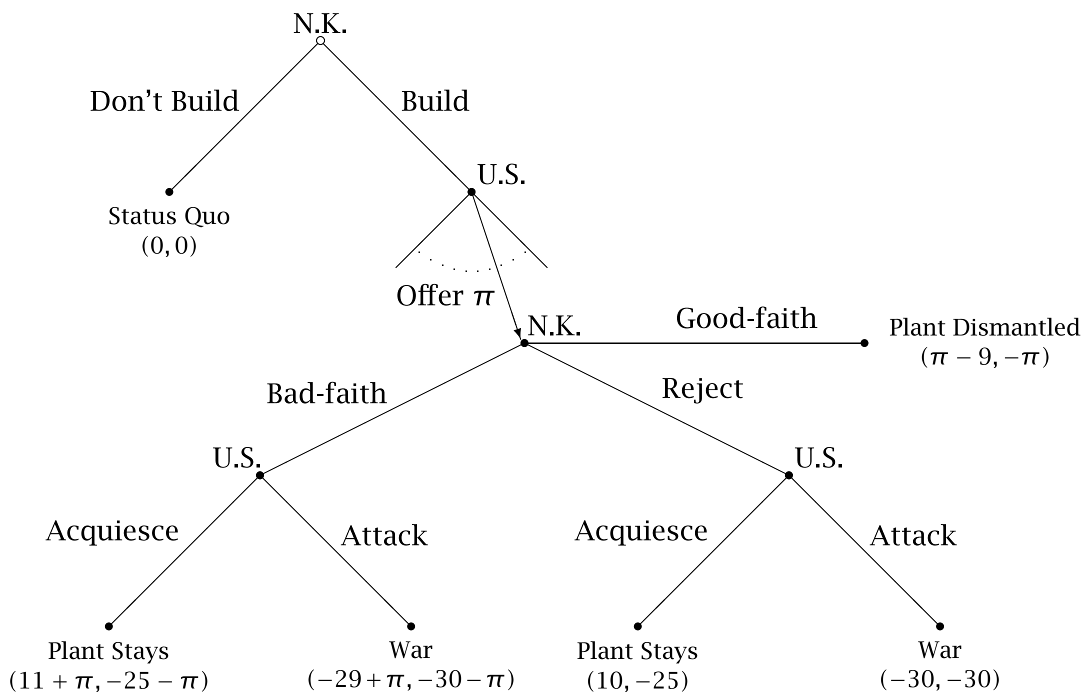

Figure 6: Bribing the North Koreans.

(a) What is the outcome in the subgame perfect equilibrium of this game?

(b) How does that differ from the outcome we arrived at in class? Why?

Answer. Although this game is very similar to the one we did in class, you should pay extra attention to the backward induction.

(a) The U.S. prefers to acquiesce both whenever N.K. rejects a bribe and whenever it negotiates in bad faith (this holds regardless of π because $-25 - \pi > -30 - \pi$ for all $\pi > 0$. Given that the U.S. acquiesces, N.K. prefers to negotiate in bad faith to rejecting an offer outright, again regardless of π because $11 + \pi > 10$ for all $\pi > 0$. It also always prefers to negotiate in bad faith to negotiating in good faith. This is because $11 + \pi > \pi - 9$, again regardless of π. Given that N.K. would negotiate in bad faith whatever it offers, the U.S. prefers to make the smallest possible bribe (which, according to our assumptions is $1 million). Given that the U.S. offers $1 million (which N.K. accepts but without dismantling the plant), N.K. prefers to build the plant.

The subgame perfect equilibrium is the pair of strategies where N.K. builds the plant and negotiates in bad faith regardless of the bribe; and where U.S. offers the smallest possible bribe ($1 million) and acquiesces whether N.K. negotiated in bad faith or simply rejected the bribe.

The outcome is that N.K. builds the plant, the U.S. makes its minimal offer, which is accepted by N.K., which then fails to dismantle the plant.

(b) This item differs from the one arrived at in class. Even though the U.S. still does not have a credible threat to use force whether or not N.K. rejected an offer or negotiated in bad faith, N.K. is unable to extract big concessions from the U.S. because it cannot make a credible promise not to negotiate in bad faith. N.K.'s inability to credibly commit to negotiating n good faith is detrimental to N.K.'s bargaining position even though the U.S. still does not have a credible threat.